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Event-Triggered Fuzzy Filtering for Nonlinear Dynamic Systems via Reduced-Order Appraoch

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***Abstract*—This paper is concerned with the problem of generalized $\mathcal{H}_2$ reduced-order filter design for continuous Takagi-Sugeno fuzzy systems using an event-triggered scheme. For a continuous Takagi-Sugeno fuzzy dynamic system, we want to establish a reduced-order filter to transform the original model into a linear lower-order one. This filter can also approximate the original system with $\mathcal{H}_2$ performance, with a new type of event-triggered scheme used to decrease the communication loads and computation resources within the network. By transforming the filtering problem to a convex optimization one, conditions are presented to design the fuzzy reduced-order filter. Finally, two illustrative examples are used to verify the feasibility and applicability of the proposed design scheme.**

***Index Terms*—Fuzzy systems, Fuzzy filter, $\mathcal{H}_2$ filtering, Reduced-order approach**

I. Introduction

Filtering on dynamical systems has received considerable attention in the past many years as its capability to estimate system states when the systems have noisy inputs. A great number of results have been obtained which are applicable to practical systems [2], [19]. Significant effort has been expended on estimation techniques, with the Kalman filtering method [10], a particularly common approach to minimize estimation errors. Kalman filtering can effectively process a dynamic system if the system consists of an explicitly known model, and the external disturbance is Gaussian noise. In actual situations, however, the exact mathematical system model is often not available, and there may be various external disturbances and system modeling errors. Therefore, many other significant and effective approaches have been developed [20], [29]. Some examples include $\mathcal{H}_2$ filtering [31], $\mathcal{H}_\infty$ filtering [30], [33], mixed $\mathcal{H}_\infty$ and passive filtering [26], $\mathcal{H}_-/\mathcal{H}_\infty$ filter design for discrete T-S fuzzy system [8], filter design for polynomial fuzzy systems [6], robust observer design for unknown input T-S models [9], fault detection filter design for T-S fuzzy systems in the finite-frequency domain, which was resolved in [7], and efficient adaptive filter design for the signal processing problem, which was proposed in [16]. Of these filtering methods, two approaches are particularly common: robust $\mathcal{H}_2$ filtering [11] and $\mathcal{H}_\infty$ filtering [32]. $\mathcal{H}_2$ filters are more sensitive to modeling errors than are $\mathcal{H}_\infty$ filters, but the latter may produce large estimation errors if the original system is disturbed by white noise. However, little previous research has been conducted on the application of $\mathcal{H}_2$ optimal estimation theory to relevant filtering problem, thus providing the initial motivation for the present study.

This paper focuses on the filtering problem for Takagi-Sugeno (T-S) fuzzy systems with an event-triggered scheme. The information transmission process in networked control systems (NCSs) is conducted through a communication channel. If network measurements are sampled with a constant $h > 0$, whether the sampling signal is transferred to the filter is decided by a pre-specified event-triggered condition (ETC). When the ETC is met, the current sampled signal is promptly released to the communication channel and transmitted to the filter. Consequently, valuable computational resources and communication bandwidth can be saved during network transmission. Therefore, a dynamic system can be converted into an error-dependent time-delay system using an effective event-triggered communication scheme. For this, appropriate filters also need to be designed to guarantee system performance if a group of linear matrix inequalities is satisfied for a given threshold parameter. Filtering problems combined with ETCs have thus attracted the interests of many researchers, leading to a number of proposed event-triggered filtering approaches, including a fault detector and controller-coordinated design [25], mixed $\mathcal{H}_\infty$ and passive filtering [18], fault detection [28], $\mathcal{H}_\infty$ filtering for delayed neural networks [4], and a distributed Kalman filter [5]. Thus, event-triggered schemes play a vital role in performance analysis and system synthesis.

Nonlinear dynamic systems in various engineering fields commonly require higher-order mathematical models [13], [14], [17], [22], which may increase the difficulty and complexity of evaluating the performance and analyzing the stability of the system in question. Hence, methods for simplifying the original system with lower-order filters based on certain criteria have been a common research focus [3], [15]. The goal of reduced-order filtering is to incorporate a filter of a lower order than the original system based on specific standards. Over the last few years, many techniques have been introduced to process mathematical models by utilizing reduced-order filters, such as the $\mathcal{H}_\infty$ [1] and $\mathcal{H}_2$ methods [23]. Model order

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reduction for electronic circuit design was introduced in [27], and the authors of [24] investigated model approximation for switched systems with stochastic perturbation. Reduced-order filter design has the advantage of flexibility and simplicity in implementation for practical applications, but it is worth noting that limited research has been conducted on $\mathcal{H}_2$ filtering for T-S fuzzy systems, particularly regarding reduced-order filtering. As a consequence, research on reduced-order filter design for T-S fuzzy systems is important in terms of both theory and practice, which has motivated us to conduct the current study.

Few studies have been conducted on $\mathcal{H}_2$ filtering and event-triggered schemes for T-S fuzzy systems, and there have been few attempts to address the related reduced-order filtering problem despite its theoretical and practical significance. Thus, the objective of this work is to resolve the problem of reduced-order $\mathcal{H}_2$ filtering for nonlinear T-S fuzzy systems that employ an event-triggered communication scheme. The main contributions of this paper are summarized below:

1) Ideal solutions are obtained for the filtering of continuous T-S fuzzy systems. Problems surrounding $\mathcal{H}_2$ filtering and reduced-order filter design are also resolved for these systems.
2) An event-triggered communication scheme, which can be used to decide whether the sampling signal will be transmitted, is coupled with a fuzzy filter to decrease the use of limited network resources.
3) Because the filter conditions include nonlinear matrix inequalities, the non-convex feasibility issue is converted to a convex optimization issue using the reciprocally convex method and readily solved by some standard available software.

II. System Description and Problem Statement

A. Model Description

Consider a class of continuous-time nonlinear systems as the target plants, which can be expressed approximately as the T-S fuzzy model below:

♦ Plant Form:

Rule i : IF $\zeta_1(t)$ is ϑ_{i1}, $\zeta_2(t)$ is ϑ_{i2}, $\ldots$, and $\zeta_p(t)$ is ϑ_{ip}, THEN

$$\begin{cases} \dot{x}(t) \;=\; A_i x(t) + B_i \omega(t), \\ y(t) \;=\; C_i x(t), \\ z(t) \;=\; E_i x(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector of the system ; $\omega(t) \in \mathbb{R}^q$ denotes the exogenous disturbance input which is assumed to belong to $\mathcal{L}_2[0,\infty)$; $y(t) \in \mathbb{R}^p$ denotes the measurement output; and $z(t) \in \mathbb{R}^m$ denotes the signal to be estimated. A_i, B_i, C_i, and E_i are system matrices of suitable dimensions. The fuzzy basis functions can be presented as

$$h_i\big(\zeta(t)\big) = \frac{v_i(\zeta(t))}{\sum\limits_{i=1}^{r} v_i(\zeta(t))}, \quad v_i\big(\zeta(t)\big) = \prod_{j=1}^{p} \vartheta_{ij}\big(\zeta_j(t)\big),$$

where $\zeta(t) = \Big[\zeta_1(t), \zeta_2(t), \cdots, \zeta_p(t)\Big]$ represents the premise variable vector, and $\vartheta_{ij}\big(\zeta_j(t)\big)$ stands for the grade of membership of $\zeta_j(t)$ in ϑ_{ij}. ϑ_{ij} represents the fuzzy sets, where $i = 1, 2, \cdots, r$, and the scalar r denotes the number of IF-THEN rules. For all t, assume that $v_i(\zeta(t)) \geqslant 0$, $\sum\limits_{i=1}^{r} v_i\big(\zeta(t)\big) > 0$. Consequently, we conclude that $h_i\big(\zeta(t)\big) \geqslant 0$, and $\sum\limits_{i=1}^{r} h_i\big(\zeta(t)\big) = 1$. Thus, the T-S fuzzy system (1) can be further rewritten in a more complete representation:

$$\begin{cases} \dot{x}(t) = \sum\limits_{i=1}^{r} h_i\big(\zeta(t)\big)\Big\{A_i x(t) + B_i \omega(t)\Big\}, \\ y(t) = \sum\limits_{i=1}^{r} h_i\big(\zeta(t)\big) C_i x(t), \\ z(t) = \sum\limits_{i=1}^{r} h_i\big(\zeta(t)\big) E_i x(t). \end{cases} \tag{2}$$

B. Structure of Reduced-Order Fuzzy Filter

We are on the stage to propose the reduced-order fuzzy filter in this subsection. Suppose the premise variable of the fuzzy model $\zeta(t)$ is obtainable for feedback, which signifies that $h_i\big(\zeta(t)\big)$ is obtainable for feedback. Assume the premise variables and membership functions of the fuzzy filter are identical to these in the T-S fuzzy plant. By utilizing the parallel distributed compensation approach, the fuzzy-rule-dependent filter is put forward to employ identical IF-THEN sections. Then, the aim is how to design a reduced-order dynamic filter described by

♦ Filter Form:

Rule i : IF $\zeta_1(t)$ is ϑ_{i1}, $\ldots$, and $\zeta_p(t)$ is ϑ_{ip}, THEN

$$\begin{cases} \dot{x}_f(t) \;=\; A_{fi} x_f(t) + B_{fi} \hat{y}(t), \\ z_f(t) \;=\; E_{fi} x_f(t), \end{cases} \tag{3}$$

where $x_f(t) \in \mathbb{R}^l$ denotes the state vector of filter with $l \leq n$; $\hat{y}(t)$ denotes practical input signal of filter; $z_f(t) \in \mathbb{R}^m$ denotes the estimate signal of $z(t)$; A_{fi}, B_{fi}, and E_{fi} are matrices with appropriate dimensions to be determined. Furthermore, the fuzzy filter in (3) is compactly represented by

$$\begin{cases} \dot{x}_f(t) = \sum\limits_{i=1}^{r} h_i\big(\zeta(t)\big)\Big\{A_{fi} x_f(t) + B_{fi} \hat{y}(t)\Big\}, \\ z_f(t) = \sum\limits_{i=1}^{r} h_i\big(\zeta(t)\big)\Big\{E_{fi} x_f(t)\Big\}. \end{cases} \tag{4}$$

C. Event-Triggered Technique

A fresh event-triggered communication technique is employed to decide whether or not the output signal $y(t)$ can be transmitted to the reduced-order filter, which can reduce the data transmission pressure and save the precious communication resources.

Employing the sampler and zero-order holder (ZOH) in communication network, the current sampled signal $y(t_kT + nT)$ will be transferred if the threshold condition satisfies as follow:

$$\Big[y(t_kT+nT) - y(t_kT)\Big]^T \Lambda_1 \Big[y(t_kT+nT) - y(t_kT)\Big] \leqslant \delta y^T(t_kT) \Lambda_2 y(t_kT), \tag{5}$$

where $\delta \in [0,1)$, Λ_1 and Λ_2 are positive definite symmetric matrices, $y(t_kT+nT)$ is the present sampling signal, and $y(t_kT)$ is the latest transmitted packet.

Define the transmission error $e_k\big(s_kT\big)$ between the latest sampled signal and the current sampled signal as

$$e_k\big(s_kT\big)=\big[y(t_kT+nT)\big]-y\big(t_kT\big).$$

Denote $s_kT=t_kT+nT$, which means the sampling time between the current instant t_kT and the approaching instant $t_{k+1}T$. Thus, the transmission error can be rewritten as

$$e_k\big(s_kT\big)=y\big(s_kT\big)-y\big(t_kT\big). \tag{6}$$

Considering event-triggered communication scheme, the T-S fuzzy system in (2) can be converted into a new system with time-delay. Assumed that q is a finite positive integer, and it satisfies $t_{k+1}=t_k+q+1$. Therefore, the time interval of ZOH is given by

$$\big[t_kT+\tau_{t_k},t_{k+1}T+\tau_{t_{k+1}}\big)=\bigcup_{n=0}^{q}\Lambda_{n,k},$$

where

$$\Lambda_{n,k}\triangleq\big[t_kT+nT+\tau_{i_k+n},t_kT+(n+1)T+\tau_{i_k+n+1}\big),\\ n=0,1,\ldots,q.$$

In this work, we define the network delay as

$$h(t)=t-\big(t_kT+nT\big)=t-s_kT,\ \ 0\leqslant h(t)\leqslant T+\bar{h}=h,$$

where $t\in\Lambda_{n,k}$. Therefore, the new ETC is inferred as

$$e_k^T\big(s_kT\big)\Lambda_1e_k\big(s_kT\big)\ \leqslant\ \delta y^T\big(t_kT\big)\Lambda_2y\big(t_kT\big). \tag{7}$$

Considering the behavior of ZOH, the input of the reduced-order filter can be further formulated as

$$\begin{aligned}\hat{y}(t)&=y(t_kT)=y(s_kT)-e_k(s_kT)\\&=y\big(t-h(t)\big)-e_k\big(t-h(t)\big),\\ t&\in\big[t_kT+\tau_{t_k},t_{k+1}T+\tau_{t_{k+1}}\big).\end{aligned} \tag{8}$$

Substituting (8) to (4), it follows that

$$\begin{cases}\dot{x}_f(t)=\displaystyle\sum_{i=1}^{r}h_i\big(\zeta(t)\big)\Big\{A_{fi}x_f(t)+B_{fi}\big[y(t-h(t)\\ \qquad -e_k(t-h(t))\big]\Big\},\\ z_f(t)=\displaystyle\sum_{i=1}^{r}h_i\big(\zeta(t)\big)\Big\{E_{fi}x_f(t)\Big\}.\end{cases} \tag{9}$$

Therefore, the fuzzy filter in (4) is transformed into a delayed-time system with the event-triggered technique described above.

Remark 1. *Because the effect of the correspondence network in traditional filter design is considered negligible, $\hat{y}(t)=y(t)$. However, the time delays within the network affect the analysis of system performance, which are considered in this article, hence $\hat{y}(t)\neq y(t)$.*

Remark 2. *The zero-order holder (ZOH), which can select the most recent data signal, is used to drive the proposed filter. Therefore, out-of-order data packets can be dropped. Data packet losses and communication network delays can be attributed to the delays caused by the network.*

D. Problem Formulation

Extending the model in (2) to contain the filter signals in (9) and the ETC in (7), the corresponding filtering error system can be obtained as

$$\begin{cases}\dot{\xi}(t)=\displaystyle\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\big(\zeta(t)\big)h_j\big(\zeta(t)\big)\Big\{\widetilde{A}_0\xi(t)+\widetilde{A}_1x(t-h(t))\\ \qquad +\widetilde{B}_0w(t)+\widetilde{C}_1e_k\big(t-h(t)\big)\Big\},\\ e_f(t)=\displaystyle\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\big(\zeta(t)\big)h_j\big(\zeta(t)\big)\Big\{\widetilde{E}_1\xi(t)\Big\},\end{cases} \tag{10}$$

where $\xi(t)\triangleq\begin{bmatrix}x(t)\\x_f(t)\end{bmatrix}$, $e_f(t)\triangleq z(t)-z_f(t)$.

$$\widetilde{A}_0=\begin{bmatrix}A_i&0\\0&A_{fj}\end{bmatrix},\widetilde{A}_1=\begin{bmatrix}0\\B_{fj}C_i\end{bmatrix},\widetilde{B}_0=\begin{bmatrix}B_i\\0\end{bmatrix},\\ \widetilde{C}_1=\begin{bmatrix}0\\-B_{fj}\end{bmatrix},\widetilde{E}_1=\begin{bmatrix}E_i&-E_{fj}\end{bmatrix},E\triangleq\begin{bmatrix}I&0\end{bmatrix}.$$

The structure of the overall filtering error system is shown in Fig. 1.

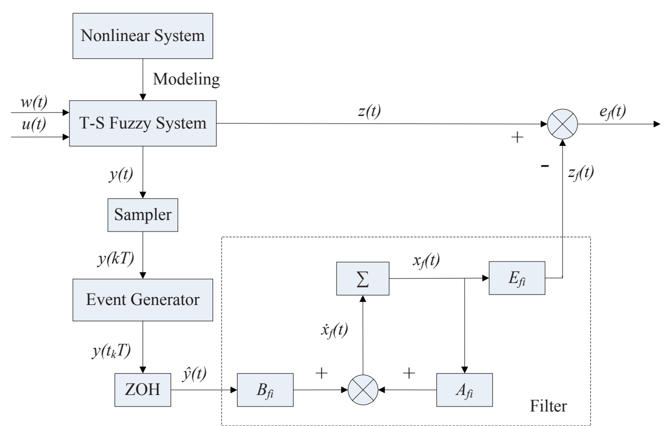

Fig. 1: Structure of the fuzzy filter with event-triggered scheme

Definition 1. *The equilibrium $\xi^*=0$ of the fuzzy filtering system (10) with $w(t)=0$ is asymptotically stable if the following condition is satisfied:*

$$\lim_{t\to\infty}\mathcal{E}\Big\{\|\xi(t)\|^2=0\Big\}=0.$$

Definition 2. *For a given scalar $\gamma>0$, the overall filtering system (10) is asymptotically stable with a generalized $\mathcal{H}_2$ disturbance attenuation level γ, if it is asymptotically stable when $w(t)=0$, and the zero initial state is set as $\xi(0)=0$,*

$$\|e(t)\|_\infty<\gamma\|w(t)\|_2,$$

for whole non-zero $w(t)\in\mathcal{L}_2[0,\infty)$, where

$$\|e(t)\|_\infty\triangleq\sup_t\sqrt{|e(t)^2|}.$$

III. Main Results

In this section, the asymptotic stability with a generalized $\mathcal{H}_2$ performance level γ of system (10) is discussed in Theorem 1, and a fuzzy reduced-order filer is designed in Theorem 2.

A. Generalized $\mathcal{H}_2$ Filtering Analysis

Sufficient conditions to ensure system (10) is asymptotic stability with an $\mathcal{H}_2$ performance are presented in the following theorem.

Theorem 1. *Given scalars $\delta \in [0,1)$, $h > 0$, $\gamma > 0$, system in (10) is asymptotically stable with a generalised $\mathcal{H}_2$ disturbance attenuation level γ if there exist matrices $P > 0$, $Q > 0$, $R > 0$, $S > 0$, $T > 0$, $\Lambda_1 > 0$, $\Lambda_2 > 0$, χ, such that, for all i, j = 1, 2, ..., r,*

$$\frac{2}{r-1}\Omega^{ii} + \Omega^{ij} + \Omega^{ji} < 0, \tag{11}$$

$$\Omega^{ii} < 0, \tag{12}$$

$$\begin{bmatrix} -P & \widetilde{E}_1^T \\ \star & -\gamma^2 I \end{bmatrix} < 0, \tag{13}$$

$$\begin{bmatrix} \mathcal{I}\otimes R + \mathcal{I}\otimes S & \chi \\ \star & \mathcal{I}\otimes R + \mathcal{I}\otimes T \end{bmatrix} > 0, \tag{14}$$

where $\Omega^{ii} = \begin{bmatrix} \Omega_1^{ii} & \Omega_2^{ii} \\ \star & \Omega_4^{ii} \end{bmatrix}, \mathcal{I} \triangleq diag\{I, 3I, 5I\}$,

$$\Omega_1^{ii} \triangleq \begin{bmatrix} \Omega_{11}^{ii} & \Omega_{12}^{ii} & \Omega_{13}^{ii} & \Omega_{14}^{ii} \\ \star & -Q-6T-9R & \Omega_{23}^{ii} & \Omega_{24}^{ii} \\ \star & \star & \Omega_{33}^{ii} & \Omega_{34}^{ii} \\ \star & \star & \star & \Omega_{44}^{ii} \end{bmatrix},$$

$$\Omega_2^{ii} \triangleq \begin{bmatrix} \Omega_{15}^{ii} & \Omega_{16}^{ii} & \Omega_{17}^{ii} & \Omega_{18}^{ii} & P^T\widetilde{C}_1 \\ 18T+36R & \Omega_{26}^{ii} & \Omega_{27}^{ii} & 0 & 0 \\ \Omega_{35}^{ii} & \Omega_{36}^{ii} & \Omega_{37}^{ii} & 0 & -\delta C_i^T\Lambda_2 \\ \Omega_{45}^{ii} & \Omega_{46}^{ii} & \Omega_{47}^{ii} & 0 & 0 \end{bmatrix},$$

$$\Omega_4^{ii} \triangleq \begin{bmatrix} \Omega_{55}^{ii} & \Omega_{56}^{ii} & \Omega_{57}^{ii} & 0 & 0 \\ \star & \Omega_{66}^{ii} & \Omega_{67}^{ii} & 0 & 0 \\ \star & \star & \Omega_{77}^{ii} & 0 & 0 \\ \star & \star & \star & \Omega_{88}^{ii} & 0 \\ \star & \star & \star & \star & \delta\Lambda_2 - \Lambda_1 \end{bmatrix},$$

with

$$\Omega_{11}^{ii} \triangleq sym\{\widetilde{A}_0^T P\} + E^T[Q - 6S - 9R + h^2 A_i^T R A_i + \frac{1}{2}h^2 A_i^T (S+T) A_i]E,$$

$$\Omega_{12}^{ii} \triangleq E^T\Big\{\sum_{m=1}^{3}(\chi_{1m}^T - \chi_{2m}^T + \chi_{3m}^T)\Big\},$$

$$\Omega_{13}^{ii} \triangleq P^T\widetilde{A}_1 + E^T\Big\{\sum_{m=1}^{3}(-\chi_{1m}^T - \chi_{2m}^T - \chi_{3m}^T) + 3R\Big\},$$

$$\Omega_{14}^{ii} \triangleq E^T\{-6S - 24R\}, \Omega_{17}^{ii} \triangleq E^T\Big\{\sum_{m=1}^{3} 6\chi_{3m}^T\Big\},$$

$$\Omega_{15}^{ii} \triangleq E^T\Big\{\sum_{m=1}^{3}(2\chi_{2m}^T - 6\chi_{3m}^T)\Big\}, \Omega_{16}^{ii} \triangleq E^T\{12S + 30R\},$$

$$\Omega_{18}^{ii} \triangleq E^T\big[h^2 A_i^T R B_i + \frac{1}{2}h^2 A_i^T (S+T) B_i\big] + P^T\widetilde{B}_0,$$

$$\Omega_{23}^{ii} \triangleq (-1)^m \sum_{m=1}^{3}(\chi_{m1} - \chi_{m2} + \chi_{m3}) + 3R,$$

$$\Omega_{24}^{ii} \triangleq (-1)^m \sum_{m=1}^{3}(2\chi_{m2} - 6\chi_{m3}),$$

$$\Omega_{26}^{ii} \triangleq (-1)^m \sum_{m=1}^{3} 6\chi_{m3}, \Omega_{27}^{ii} \triangleq -12T - 30R,$$

$$\Omega_{33}^{ii} \triangleq sym\Big\{\sum_{m=1}^{3}(\chi_{m1} - \chi_{m2} + \chi_{m3})\Big\} - 6S - 6T - 18R + \delta C_i^T \Lambda_2 C_i,$$

$$\Omega_{34}^{ii} \triangleq \sum_{m=1}^{3}(2\chi_{m2} - 6\chi_{m3}) + 18T + 36R,$$

$$\Omega_{35}^{ii} \triangleq (-1)^m \sum_{m=1}^{3}(2\chi_{2m}^T - 6\chi_{3m}^T) - 6S - 24R,$$

$$\Omega_{36}^{ii} \triangleq \sum_{m=1}^{3} 6\chi_{m3} - 12T - 30R,$$

$$\Omega_{37}^{ii} \triangleq (-1)^m \sum_{m=1}^{3} 6\chi_{3m}^T + 12S + 30R,$$

$$\Omega_{44}^{ii} \triangleq -18S - 66T - 192R, \quad E \triangleq \begin{bmatrix} I & 0 \end{bmatrix},$$

$$\Omega_{45}^{ii} \triangleq -4\chi_{22}^T + 12\chi_{23}^T + 12\chi_{32}^T - 36\chi_{33}^T,$$

$$\Omega_{46}^{ii} \triangleq 24S + 48T + 180R, \Omega_{47}^{ii} \triangleq -12\chi_{32}^T + 36\chi_{33}^T,$$

$$\Omega_{55}^{ii} \triangleq -18S - 66T - 192R, \Omega_{56}^{ii} \triangleq -12\chi_{23} + 36\chi_{33},$$

$$\Omega_{57}^{ii} \triangleq 24S + 48T + 180R, \Omega_{66}^{ii} \triangleq -36S - 36T - 180R,$$

$$\Omega_{67}^{ii} \triangleq -36\chi_{33}^T, \Omega_{77}^{ii} \triangleq -36S - 36T - 180R,$$

$$\Omega_{88}^{ii} \triangleq h^2 B_i^T R B_i + \frac{1}{2}h^2 B_i^T (S+T) B_i - I.$$

Proof: Construct the following Lyapunov function candidate

$$V(t) = \sum_{m=1}^{3} V_m(t), \tag{15}$$

where

$$\begin{aligned}
V_1(t) &\triangleq \xi^T(t) P \xi(t), \\
V_2(t) &\triangleq \int_{t-h}^{t} \xi^T(s) Q \xi(s) ds \\
&\quad + h\int_{-h}^{0}\int_{t+\theta}^{t} \dot{\xi}^T(s) R \dot{\xi}(s) ds d\theta, \\
V_3(t) &\triangleq \int_{-h}^{0}\int_{r}^{0}\int_{t+\theta}^{t} \dot{\xi}^T(s) S \dot{\xi}(s) ds d\theta dr \\
&\quad + \int_{-h}^{0}\int_{-h}^{r}\int_{t+\theta}^{t} \dot{\xi}^T(s) T \dot{\xi}(s) ds d\theta dr.
\end{aligned}$$

By taking the derivation of $V(t)$, it follows that

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t)$$

$$
\begin{aligned}
&\leqslant sym\Big\{\xi^T(t)\widetilde{A}_0^T P\xi(t) + \xi^T(t-h(t))\widetilde{A}_1^T P\xi(t) \\
&\quad +w^T(t)\widetilde{B}_0^T P\xi(t) + e_k^T(t-h(t))\widetilde{C}_1^T P\xi(t)\Big\} \\
&\quad +\xi^T(t)Q\xi(t) - \xi^T(t-h)Q\xi(t-h) \\
&\quad +\frac{1}{2}h^2\dot{\xi}^T(t)(S+T)\dot{\xi}(t) + h^2\dot{\xi}^T(t)R\dot{\xi}(t) \\
&\quad -2\Gamma_7^T S\Gamma_7 - 4\Gamma_8^T S\Gamma_8 - 2\Gamma_9^T S\Gamma_9 - 4\Gamma_{10}^T S\Gamma_{10} \\
&\quad -2\Gamma_{11}^T T\Gamma_{11} - 2\Gamma_{12}^T T\Gamma_{12} - 4\Gamma_{13}^T T\Gamma_{13} \\
&\quad -4\Gamma_{14}^T T\Gamma_{14} - \Gamma_{1,6}^T \Xi\Gamma_{1,6}
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
&\Gamma_1 \triangleq \Upsilon_3 - \Upsilon_2, \quad \Gamma_2 \triangleq \Upsilon_3 + \Upsilon_2 - 2\Upsilon_5, \\
&\Gamma_3 \triangleq \Upsilon_3 - \Upsilon_2 + 6\Upsilon_5 - 6\Upsilon_7, \quad \Gamma_4 \triangleq \Upsilon_1 - \Upsilon_3, \\
&\Gamma_5 \triangleq \Upsilon_1 + \Upsilon_3 - 2\Upsilon_4, \quad \Gamma_6 \triangleq \Upsilon_1 - \Upsilon_3 + 6\Upsilon_4 - 6\Upsilon_6, \\
&\Gamma_7 \triangleq \Upsilon_3 - \Upsilon_5, \quad \Gamma_8 \triangleq \Upsilon_3 + 2\Upsilon_5 - 3\Upsilon_7, \\
&\Gamma_9 \triangleq \Upsilon_1 - \Upsilon_4, \quad \Gamma_{11} \triangleq \Upsilon_2 - \Upsilon_5, \\
&\Gamma_{10} \triangleq \Upsilon_1 + 2\Upsilon_4 - 3\Upsilon_6, \quad \Gamma_{12} \triangleq \Upsilon_3 - \Upsilon_4, \\
&\Gamma_{13} \triangleq \Upsilon_2 - 4\Upsilon_5 + 3\Upsilon_7, \quad \Gamma_{14} \triangleq \Upsilon_3 - 4\Upsilon_4 + 3\Upsilon_6, \\
&\Gamma_{1,6} \triangleq \left[\begin{array}{cccccc} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Gamma_5 & \Gamma_6 \end{array}\right], \\
&\Xi \triangleq \left[\begin{array}{cc} diag\{R,3R,5R\} & \chi \\ \star & diag\{R,3R,5R\}\} \end{array}\right], \\
&\Upsilon(t) \triangleq col\Big\{\xi(t),\ \xi(t-h),\ \xi(t-h(t)),\ \frac{1}{h(t)}\int_{t-h(t)}^{t}\xi(\alpha)d\alpha, \\
&\quad \frac{1}{h-h(t)}\int_{t-h}^{t-h(t)}\xi(\alpha)d\alpha,\ \frac{2}{h^2(t)}\int_{-h(t)}^{0}\int_{t+\beta}^{t}\xi(\alpha)d\alpha d\beta, \\
&\quad \frac{2}{(h-h(t))^2}\int_{-h}^{-h(t)}\int_{t+\beta}^{t-h(t)}\xi(\alpha)d\alpha d\beta, w(t), \\
&\quad e_k(t-h(t))\ \Big\}.
\end{aligned}
$$

For the sake of completing the establishment of the generalised $\mathcal{H}_2$ performance for the overall filtering system in (10), the zero initial state is assumed as $\xi(0)=0$, so $V(\xi(t))|_{t=0}=0$. Consider the following induced index

$$
\begin{aligned}
\mathcal{J}(t) &\triangleq \int_0^\infty -w^T(t)w(t)dt \\
&\leqslant \Big[\int_0^\infty -w^T(t)w(t)dt\Big] + V(t) - V(0) \\
&\triangleq \int_0^\infty \Big[-w^T(t)w(t) + \dot{V}(t)\Big]dt
\end{aligned} \tag{17}
$$

Considering the event-triggered mechanism in (7), we denote

$$
\Delta(t) = \delta y^T(t_kT)\Lambda_2 y(t_kT) - e_k^T(s_kT)\Lambda_1 e_k(s_kT) > 0. \tag{18}
$$

It follows from (16) to (18) that

$$
\begin{aligned}
-w^T(t)w(t) + \dot{V}(t) + \Delta(t) \leqslant & \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\zeta(t))h_j(\zeta(t)) \\
& \Big\{\Psi^T(t)\Omega^{ii}\Psi(t)\Big\}.
\end{aligned} \tag{19}
$$

Then, in view of the condition in (12), it is easy to get $\Omega^{ii} < 0$. From the condition in (18) and (19), it follows that

$$
-w^T(t)w(t) + \dot{V}(t) < 0 \tag{20}
$$

Through the integral calculation on the two sides of (20) from 0 to ∞, we get

$$
V(t) < \int_0^\infty w^T(t)w(t)dt \tag{21}
$$

Furthermore, the inequality in (13) can be converted into the following inequality by the Schur Complement method:

$$
\widetilde{E}_1^T\widetilde{E}_1 - \gamma^2 P < 0 \tag{22}
$$

Thus, it can be easily established that for all $t > 0$,

$$
\begin{aligned}
e_f^T(t)e_f(t) - \gamma^2 V_1(t) \leqslant & \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\zeta(t))h_j(\zeta(t)) \\
& \Big\{\xi^T(t)\big(\widetilde{E}_1^T\widetilde{E}_1 - \gamma^2 P\big)\xi(t)\Big\}.
\end{aligned} \tag{23}
$$

From (22), we can easily conclude that

$$
e_f^T(t)e_f(t) < \gamma^2 V_1(t) < \gamma^2 V(t). \tag{24}
$$

It follows from (21) and (24) that for all $t > 0$

$$
e_f^T(t)e_f(t) < \gamma^2 V(t) < \gamma^2 \int_0^\infty w^T(t)w(t)dt. \tag{25}
$$

Getting the supremum over $t > 0$ leads to $\|e(t)\|_\infty < \gamma\|w(t)\|_2$ for all non-zero $w(t) \in \mathcal{L}_2[0,\infty)$, thus it completes the proof. ■

B. Reduced-order Filter Design

The following result provides conditions to solve the $\mathcal{H}_2$ filtering problem for system (10) by linearization method.

Theorem 2. *Given scalars $\delta \in [0,1)$, $h > 0$, $\gamma > 0$, if there exist real matrices $\mathbb{W} > 0$, $P_1 > 0$, $Q > 0$, $R > 0$, $S > 0$, $T > 0$, $\Lambda_1 > 0$, $\Lambda_2 > 0$, χ, and the parameters of the filter $\widetilde{A}_{fi}, \widetilde{B}_{fi}, \widetilde{E}_{fi}$ such that the following matrix inequations are satisfied for $i,j = 1,2,\ldots,r$:*

$$
\frac{2}{r-1}\Psi^{ii} + \Psi^{ij} + \Psi^{ji} < 0, \tag{26}
$$

$$
\Psi^{ii} < 0, \tag{27}
$$

$$
\left[\begin{array}{ccc} -P_1 & -\mathcal{H}\mathbb{W} & E_i^T \\ \star & -\mathbb{W}^T & -\widetilde{E}_{fi}^T \\ \star & \star & -\gamma^2 I \end{array}\right] < 0, \tag{28}
$$

where $\Psi^{ii} = \left[\begin{array}{cc} \Psi_1^{ii} & \Psi_2^{ii} \\ \star & \Psi_4^{ii} \end{array}\right]$,

$$
\Psi_1^{ii} \triangleq \left[\begin{array}{ccccc}
\Psi_{111}^{ii} & \Psi_{112}^{ii} & \Psi_{12}^{ii} & \Psi_{13}^{ii} & \Psi_{14}^{ii} \\
\star & \Psi_{122}^{ii} & 0 & \bar{\Psi}_{13}^{ii} & 0 \\
\star & \star & \Omega_{22}^{ii} & \Omega_{23}^{ii} & \Omega_{24}^{ii} \\
\star & \star & \star & \Omega_{33}^{ii} & \Omega_{34}^{ii} \\
\star & \star & \star & \star & \Omega_{44}^{ii}
\end{array}\right],
$$

$$
\Psi_2^{ii} \triangleq \left[\begin{array}{ccccc}
\Psi_{15}^{ii} & \Psi_{16}^{ii} & \Psi_{17}^{ii} & \Psi_{18}^{ii} & \Psi_{19}^{ii} \\
0 & 0 & 0 & \bar{\Psi}_{18}^{ii} & \bar{\Psi}_{19}^{ii} \\
\Omega_{25}^{ii} & \Omega_{26}^{ii} & \Omega_{27}^{ii} & 0 & 0 \\
\Omega_{35}^{ii} & \Omega_{36}^{ii} & \Omega_{37}^{ii} & 0 & -\delta C_i^T\Lambda_2 \\
\Omega_{45}^{ii} & \Omega_{46}^{ii} & \Omega_{47}^{ii} & 0 & 0
\end{array}\right],
$$

$$
\Psi_4^{ii} \triangleq \Omega_4^{ii},
$$

with

$$\Psi_{111}^{ii} \triangleq A_i^T P_1 + P_1 A_i + Q - 6S - 9R + h^2 A_i^T R A_i + \frac{1}{2} h^2 A_i^T (S+T) A_i, \Psi_{112}^{ii} \triangleq A_i^T \mathcal{H}\mathbb{W} + \mathcal{H}\widetilde{A}_{fi},$$

$$\Psi_{122}^{ii} \triangleq \widetilde{A}_{fi} + \widetilde{A}_{fi}^T, \Psi_{12}^{ii} \triangleq \sum_{m=1}^{3} (\chi_{1m}^T - \chi_{2m}^T + \chi_{3m}^T),$$

$$\Psi_{13}^{ii} \triangleq \sum_{m=1}^{3} (-\chi_{1m}^T - \chi_{2m}^T - \chi_{3m}^T) + 3R + \mathcal{H}\widetilde{B}_{fi} C_i,$$

$$\Psi_{14}^{ii} \triangleq -6S - 24R, \Psi_{15}^{ii} \triangleq \sum_{m=1}^{3} (2\chi_{2m}^T - 6\chi_{3m}^T),$$

$$\Psi_{16}^{ii} \triangleq 12S + 30R, \Psi_{17}^{ii} \triangleq \sum_{m=1}^{3} 6\chi_{3m}^T,$$

$$\Psi_{18}^{ii} \triangleq P_1 B_i + h^2 A_i^T R B_i + \frac{1}{2} h^2 A_i^T (S+T) B_i,$$

$$\Psi_{19}^{ii} \triangleq -\mathcal{H}\widetilde{B}_{fi}, \bar{\Psi}_{13}^{ii} \triangleq \widetilde{B}_{fi} C_i$$

$$\bar{\Psi}_{18}^{ii} \triangleq \mathbb{W}^T \mathcal{H}^T B_i, \bar{\Psi}_{19}^{ii} \triangleq -\widetilde{B}_{fi},$$

$$\Omega_{22}^{ii} \triangleq -Q - 6T - 9R, \Omega_{25}^{ii} \triangleq 18T + 36R,$$

then the $\mathcal{H}_2$ reduced-order filter design problem is solvable. Furthermore, the parameters of the desired $\mathcal{H}_2$ filter in (4) are given as

$$\begin{bmatrix} A_{fi} & B_{fi} \\ E_{fi} & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{W}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{A}_{fi} & \widetilde{B}_{fi} \\ \widetilde{E}_{fi} & 0 \end{bmatrix}. \quad (29)$$

Proof: Based on Theorem 1, if the conditions (11)-(13) hold, then the nonsingular matrix P can be partitioned as

$$P \triangleq \begin{bmatrix} P_1 & \mathcal{H}P_2 \\ \star & P_3 \end{bmatrix}, \quad (30)$$

where

$$\mathcal{H} = [I_{r\times r} \quad 0_{r\times(n-r)}]^T,$$
$$P_1 \in R^{n\times n}, \quad P_2 \in R^{r\times r}, \quad P_3 \in R^{r\times r}.$$

Assume that P_2 is nonsingular, and to prove it, define

$$M \triangleq P + \varrho N (\varrho > 0)$$

and

$$N \triangleq \begin{bmatrix} 0_{n\times n} & \mathcal{H} \\ \star & 0_{r\times r} \end{bmatrix}, M \triangleq \begin{bmatrix} M_1 & \mathcal{H}M_2 \\ \star & M_3 \end{bmatrix}. \quad (31)$$

Owing to $P > 0$, we can easily obtain that $M > 0$ for $\varrho > 0$. Therefore, it is simple to validate that M_2 is nonsingular for an arbitrary small $\varrho > 0$, and the equation above is feasible with P. Thus, without loss of generality, P_2 is assumed to be nonsingular subject to M_2.

Based on the above discussion, several matrices are defined as below:

$$\mathbb{U} \triangleq \begin{bmatrix} I & 0 \\ 0 & P_3^{-1}P_2^T \end{bmatrix}, \mathbb{V} \triangleq P_1, \mathbb{W} \triangleq P_2 P_3^{-1} P_2^T, \quad (32)$$

and

$$\begin{bmatrix} \widetilde{A}_{fi} & \widetilde{B}_{fi} \\ \widetilde{E}_{fi} & 0 \end{bmatrix} \triangleq \begin{bmatrix} P_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{fi} & B_{fi} \\ E_{fi} & 0 \end{bmatrix} \begin{bmatrix} P_3^{-1}P_2^T & 0 \\ 0 & I \end{bmatrix}. \quad (33)$$

Then, we can get

$$\mathbb{U}^T \widetilde{A}_0^T P \mathbb{U} \triangleq \begin{bmatrix} A_i^T P_1 & A_i^T \mathcal{H}\mathbb{W} \\ \widetilde{A}_{fi}^T \mathcal{H}^T & \widetilde{A}_{fi}^T \end{bmatrix},$$

$$\mathbb{U}^T P \widetilde{A}_0 \mathbb{U} \triangleq \begin{bmatrix} P_1 A_i & \mathcal{H}\widetilde{A}_{fi} \\ \mathbb{W}^T \mathcal{H}^T A_i & \widetilde{A}_{fi} \end{bmatrix},$$

$$\mathbb{U}^T P \mathbb{U} \triangleq \begin{bmatrix} P_1 & \mathcal{H}\mathbb{W} \\ \mathbb{W}^T \mathcal{H}^T & \mathbb{W}^T \end{bmatrix}, \mathbb{U}^T \widetilde{E}_1^T \triangleq \begin{bmatrix} E_i^T \\ -\widetilde{E}_{fi}^T \end{bmatrix},$$

$$\mathbb{U}^T P \widetilde{A}_1 \triangleq \begin{bmatrix} \mathcal{H}\widetilde{B}_{fi} C_i \\ \widetilde{B}_{fi} C_i \end{bmatrix}, \mathbb{U}^T P \widetilde{B}_0 \triangleq \begin{bmatrix} P_1 B_i \\ \mathbb{W}^T \mathcal{H}^T B_i \end{bmatrix},$$

$$\mathbb{U}^T P \widetilde{C}_1 \triangleq \begin{bmatrix} -\mathcal{H}\widetilde{B}_{fi} \\ -\widetilde{B}_{fi} \end{bmatrix}. \quad (34)$$

Executing congruence transformations to (11), (12), and (13) with matrices

$$\text{diag}\{ \mathbb{U} \quad I \quad I \quad I \quad I \quad I \quad I \quad I \quad I \},$$
$$\text{diag}\{ \mathbb{U} \quad I \quad I \quad I \quad I \quad I \quad I \quad I \quad I \},$$
$$\text{diag}\{ \mathbb{U} \quad I \},$$

respectively, we obtain that the equalities in (26)–(28) hold if (32)–(34) are considered. Therefore, the error system in (10) can be guaranteed to be asymptotically stable with a generalized $\mathcal{H}_2$ disturbance attenuation level γ. Moreover, note that (33) is equivalent to

$$\begin{bmatrix} A_{fi} & B_{fi} \\ E_{fi} & 0 \end{bmatrix} \triangleq \begin{bmatrix} (P_2^{-T}P_3)^{-1}\mathbb{W}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{A}_{fi} & \widetilde{B}_{fi} \\ \widetilde{E}_{fi} & 0 \end{bmatrix} \begin{bmatrix} P_2^{-T}P_3 & 0 \\ 0 & I \end{bmatrix}. \quad (35)$$

Therefore, the parameters (A_{fi}, B_{fi}, E_{fi}) in (4) can be given by (35), and it is obvious that $P_2^{-T}P_3$ can obtain the state-space form by means of similarity transformation. And in general, let $P_2^{-T}P_3 = I$, we can get (29), which can be employed to construct the reduced-order $\mathcal{H}_2$ fuzzy filter in (4). Hence, it completes the proof. ■

Remark 3. *Note that matrix $\mathcal{H}$, which is put forward in Theorem 2, plays a key role during the design process of the reduced-order filter because of its use as an order reduction factor. The result in Theorem 2 becomes a full-order case when $\mathcal{H}$ is chosen as the unit matrix, and this is simpler than using the reduced-order filtering.*

IV. Numerical Examples

In this section, two examples are provided to illustrate the effectiveness and feasibility of the reduced-order $\mathcal{H}_2$ fuzzy filter design scheme. Firstly, a numerical example is employed to manifest the validity of the proposed technique in this paper. Then a tunnel diode circuit system is approximated by a class of T-S fuzzy systems, and the desired performance index of the tunnel diode circuit system is achieved.

Example 1. *Consider the following continuous T-S fuzzy system in (2):*

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} h_i\big(\zeta(t)\big)\Big\{A_i x(t) + B_i \omega(t)\Big\}, \\ y(t) = \sum_{i=1}^{2} h_i\big(\zeta(t)\big) C_i x(t), \\ z(t) = \sum_{i=1}^{2} h_i\big(\zeta(t)\big) E_i x(t). \end{cases}$$

with

$$A_1 = \begin{bmatrix} -1.0 & 0.2 & 0.4 \\ 0.1 & -1.0 & 0.1 \\ 0.4 & 0.0 & -1.2 \end{bmatrix}, B_1 = \begin{bmatrix} 0.8 \\ 1.0 \\ 1.2 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 1.0 & 1.2 & 0.8 \end{bmatrix}, E_1 = \begin{bmatrix} 0.2 & 0.8 & 0.5 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1 & 0.2 & 0.0 \\ 0.4 & -1.1 & 0.1 \\ 0.1 & 0.3 & -1.0 \end{bmatrix}, B_2 = \begin{bmatrix} 1.2 \\ 1.0 \\ 1.0 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.6 & 1.0 & 0.6 \end{bmatrix}, E_2 = \begin{bmatrix} 0.5 & 0.2 & 0.6 \end{bmatrix}.$$

Then we strive to find effective reduced-order filters in (4) to approximate the aforementioned system with an $\mathcal{H}_2$ performance by employing convex linearization approach. Here we assume $h = 0.12$, $\delta = 0.8$.

Case 1. *First, the case of reduced-order filtering ($k = 2$) is considered. By solving the conditions in (11)–(14), we can obtain the corresponding event-triggered parameters as $\Lambda_1 = 41.4731$ and $\Lambda_2 = 0.0088$. Furthermore, the $\mathcal{H}_2$ reduced-order filter parameters are shown as below via calculating the conditions in Theorem 2:*

$$A_{f1} = \begin{bmatrix} -2.7637 & -1.4605 \\ -1.4895 & -3.0605 \end{bmatrix},$$
$$B_{f1} = \begin{bmatrix} -1.4054 \\ -1.3933 \end{bmatrix}, E_{f1} = \begin{bmatrix} -0.3407 & -0.3926 \end{bmatrix},$$
$$A_{f2} = \begin{bmatrix} -2.2015 & -1.3880 \\ -0.4678 & -3.0639 \end{bmatrix},$$
$$B_{f2} = \begin{bmatrix} -1.6967 \\ -1.6954 \end{bmatrix}, E_{f2} = \begin{bmatrix} -0.4223 & -0.2202 \end{bmatrix}.$$

Case 2. *Then we consider the reduced-order filtering problem ($k = 1$), and the corresponding event-triggered parameters can be obtained as $\Lambda_1 = 22.4914$ and $\Lambda_2 = 0.0109$. Moreover, the relevant parameters of the $\mathcal{H}_2$ reduced-order filters are calculated below:*

$$A_{f1} = \begin{bmatrix} -6.9118 \end{bmatrix}, B_{f1} = \begin{bmatrix} -0.1908 \end{bmatrix}, E_{f1} = \begin{bmatrix} -0.7967 \end{bmatrix},$$
$$A_{f2} = \begin{bmatrix} -2.1357 \end{bmatrix}, B_{f2} = \begin{bmatrix} -0.3896 \end{bmatrix}, E_{f2} = \begin{bmatrix} -0.9851 \end{bmatrix}.$$

To show the reduced-order filtering performance, the initial state of the overall filtering error system (10) is assumed to be zero, which means $x(t) = 0$, $x_f(t) = 0$. Moreover, the exogenous disturbance $w(t)$ is set as $w(t) = \sin(0.3t)\exp(-0.2t)$. Furthermore, assume that the fuzzy basis functions are chosen as

$$\begin{cases} h_1\big(\zeta(t)\big) = \dfrac{1 - \big[sin(x_1(t))\big]^2}{2}, \\ h_2\big(\zeta(t)\big) = \dfrac{1 + \big[sin(x_1(t))\big]^2}{2}. \end{cases}$$

By using efficient MATLAB toolbox, the simulations of the proposed $\mathcal{H}_2$ filters are presented in Figs. 2–7. Among them, the corresponding event-triggered release time intervals diagrams of two-order filter and one-order filter are drawn in Fig. 2 and Fig. 5, respectively. The outputs of system model (2) and reduced-order filters ($k = 2$, $k = 1$) are plotted in Fig. 3 and Fig. 6. In addition, Fig. 4 and Fig. 7 manifest the corresponding filtering errors $e_f(t)$ between them. From these figures, one can see that the $\mathcal{H}_2$ reduced-order filter via event-triggered technique realizes an ideal estimation of $z(t)$ and the valuable communication bandwidth and computation resources during transmissions can be saved to a certain degree.

Example 2. *In this example, we consider a tunnel diode circuit [12] which has been introduced in Fig. 8. The tunnel diode system is*

$$i_D(t) = 0.002 v_D(t) + 0.01 v_D^3(t).$$

Assume the state variables $x_1(t) = v_C(t)$, $x_2(t) = i_L(t)$. The circuit is governed by the following equalities:

$$\begin{cases} C\dot{x}_1(t) = -0.002x_1(t) + x_2(t) - 0.01x_1^3(t), \\ L\dot{x}_2(t) = -x_1(t) - Rx_2(t) + w(t), \\ y(t) = x_1(t), \\ z(t) = x_1(t) + w(t), \end{cases} \tag{36}$$

where $w(t)$ denotes the disturbance signal, $y(t)$ denotes the measurement output signal, and $z(t)$ denotes the signal to be evaluated. The parameters in the circuit are given as $C = 20mF$, $L = 1H$, and $R = 10\Omega$. Thus, the circuit system is further rewritten as

$$\begin{cases} \dot{x}_1(t) = -0.1x_1(t) + 50x_2(t) - [0.5x_1^2(t)] \times x_1(t), \\ \dot{x}_2(t) = -x_1(t) - 10x_2(t) + w(t), \\ y(t) = x_1(t), \\ z(t) = x_1(t) + w(t). \end{cases} \tag{37}$$

Therefore, we can obtain the two-rule T-S fuzzy model to approximate the nonlinear circuit system (37) as follows.

♦ ***Plant Form:***

***Rule** 1: IF $x_2(t)$ is $\mathcal{N}_1(x_1(t))$ THEN*

$$\begin{cases} \dot{x}(t) = A_1 x(t) + B_1 \omega(t), \\ y(t) = C_1 x(t), \\ z(t) = E_1 x(t). \end{cases}$$

***Rule** 2: IF $x_2(t)$ is $\mathcal{N}_2(x_1(t))$ THEN*

$$\begin{cases} \dot{x}(t) = A_2 x(t) + B_2 \omega(t), \\ y(t) = C_2 x(t), \\ z(t) = E_2 x(t). \end{cases}$$

On account of the above T-S fuzzy model, the relevant system parameters can be inferred as

$$A_1 = \begin{bmatrix} -0.1 & 50 \\ -1 & -10 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, E_1 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -4.6 & 50 \\ -1 & -10 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Furthermore, the membership functions employed in this example are defined as:

$$h_1\big(\zeta(t)\big) = \begin{cases} \dfrac{3 + x_1(t)}{3}, & -3 \leq x_1(t) \leq 0, \\ \dfrac{3 - x_1(t)}{3}, & 0 \leq x_1(t) \leq 3, \\ 0, & elsewhere \end{cases}$$
$$h_2\big(\zeta(t)\big) = 1 - h_1\big(\zeta(t)\big).$$

Moreover, we are committed to finding the reduced-order filter which is in the format of (4), to approximate the aforementioned system with an $\mathcal{H}_2$ performance via convex linearization approach. Under the assumption that $h = 0.06$, $\delta = 0.8$, and by employing Theorem 2, we can obtain the corresponding event-triggered matrices as $\Lambda_1 = 116.0829$ and $\Lambda_2 = 0.1179$. In addition, the relevant parameters of $\mathcal{H}_2$ reduced-order filter are calculated as

$$A_{f1} = -16.7582, \quad B_{f1} = -16.4151, \quad E_{f1} = -0.3709,$$
$$A_{f2} = -38.3097, \quad B_{f2} = -16.6927, \quad E_{f2} = -0.3709.$$

Additionally, to investigate the reduced-order filtering performance of the obtained models, the initial state is assumed to be $\xi(0) = 0$, which means $x(0) = 0$ and $x_f(0) = 0$; the external disturbance signal $w(t)$ is assumed to be $w(t) = exp(-0.1t)sin(0.1t)$, with $t \geq 0$. The simulation results in Example 2 are shown in Fig. 9–11. Among them, the event-triggered release time figure is plotted in Fig. 9. Fig. 10 draws the outputs of the system model and the aforementioned reduced-order filter. Fig. 11 draws the filtering error between them. Moreover, Fig. 9 shows that over the time interval $[0, 40s]$, just 155 sampled data packets are transmitted, leading to a transmission rate of 46.5%, which means that communication resources can be saved by 53.5%. Therefore, the findings from Figs. 9–11 demonstrate the validity of $\mathcal{H}_2$ reduced-order filter design scheme and the bandwidth usages of the network are reduced by employing the event-triggered communication scheme.

Remark 4. *Because the obtained filtering conditions involve nonlinear matrix inequalities (NLMIs), the reciprocally convex method is employed to recast the fuzzy filter design as a convex optimization problem subject to linear matrix inequalities (LMIs), which can be easily resolved with MATLAB. The conditions required for the solvability of an $\mathcal{H}_2$ reduced-order filter are presented in Theorem 2. For the preset triggered parameter δ, the corresponding allowable performance level γ, the filter parameters A_{fi}, B_{fi}, and E_{fi} and the event-triggered matrices Λ_1 and Λ_2 are designed from the derived results, which can be easily achieved by employing standard numerical software. This means that the chosen triggered parameter determines the performance of the filtering error system.*

Remark 5. *In this paper, we apply the reciprocally convex technique combined with a new Lyapunov function in (15), which contributes to the stability analysis and performance evaluation of the concerned system. The reciprocally convex method is utilized to handle the time delay, which reduces the conservativeness and the computational complexity. The event-triggered scheme is proposed to reduce the number of information transmissions while retaining ideal system performance. For the resulting filtering error system, data packets of sampled signals are transmitted by the event-trigger scheme, i.e., information transmissions are available only when the defined triggering criteria are satisfied.*

V. Conclusion

This paper addresses $\mathcal{H}_2$ reduced-order filter design for continuous fuzzy logic systems using event-triggered communication. First, the conditions required for the presence of $\mathcal{H}_2$ reduced-order fuzzy filters are derived in terms of matrix inequalities using Lyapunov-Krasovskii functions, meaning that the resulting overall filtering system is asymptotically stable in terms of $\mathcal{H}_2$. The solvability conditions for the proposed reduced-order filter are then established. A novel event-triggered technique, which can be used to decide whether the sampling signals are transmitted, is also presented to decrease the communication loads and use of computational resources within the network. Finally, the simulation results illustrate that the proposed design scheme is valid and effective. It is expected that the proposed reduced-order filter design can be extended into sampled-data T-S fuzzy systems or polynomial fuzzy systems. Applying the theoretical achievements of this work to practical complex systems such as power systems is an interesting avenue for future research.

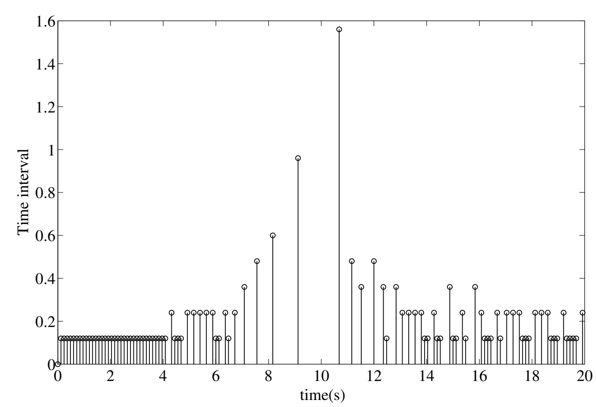

Fig. 2: The release instants and intervals in Case 1

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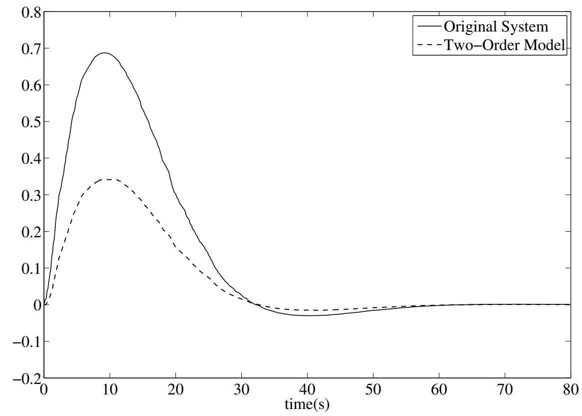

Fig. 3: Outputs $z(t)$ and its estimation $z_f(t)$ in Case 1

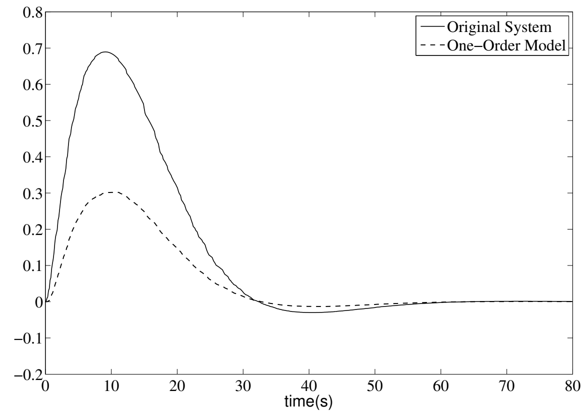

Fig. 6: Outputs $z(t)$ and its estimation $z_f(t)$ in Case 2

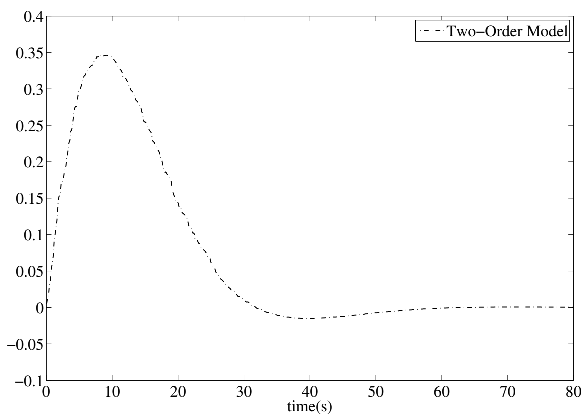

Fig. 4: Filtering errors $e_f(t)$ in Case 1

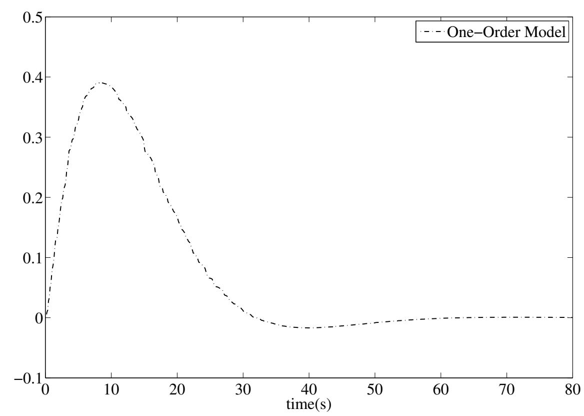

Fig. 7: Filtering errors $e_f(t)$ in Case 2

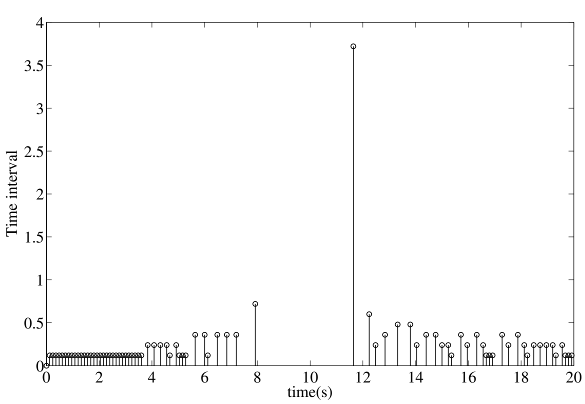

Fig. 5: The release instants and intervals in Case 2

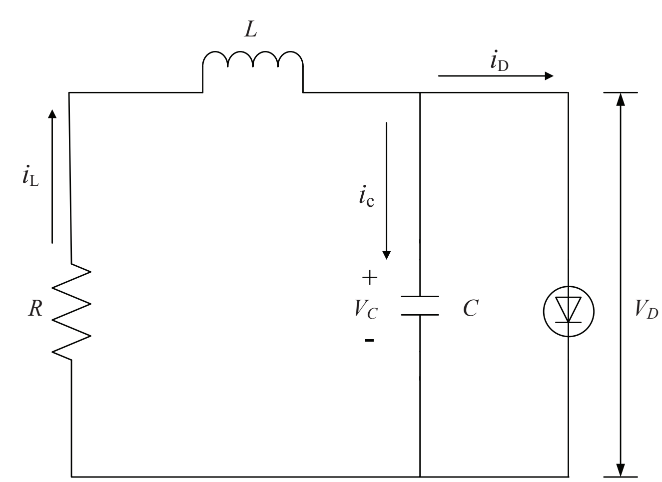

Fig. 8: Tunnel diode circuit in Example 2

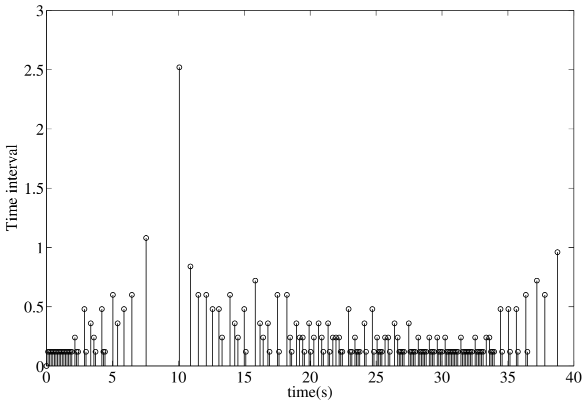

Fig. 9: The release instants and intervals in Example 2

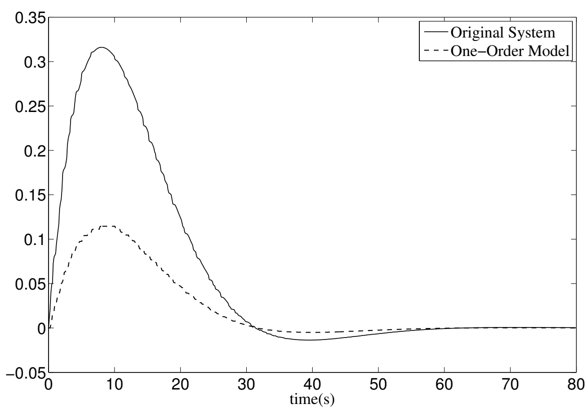

Fig. 10: Output $z(t)$ and its estimation $z_f(t)$

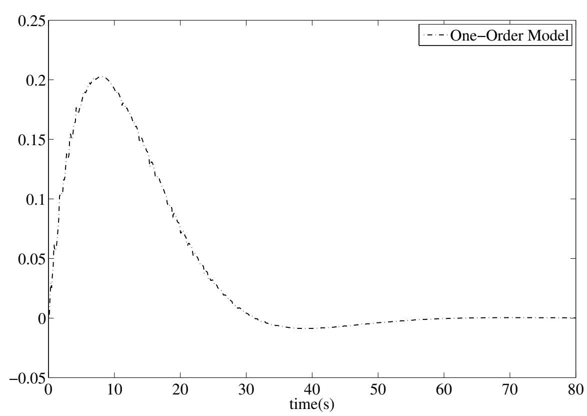

Fig. 11: Filtering errors $e_f(t)$

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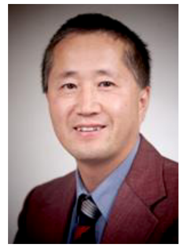

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